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THE SEMICLASSICAL EXPANSION OF THE ANHARMONIC-OSCILLATOR PROPAG--ETC(U)

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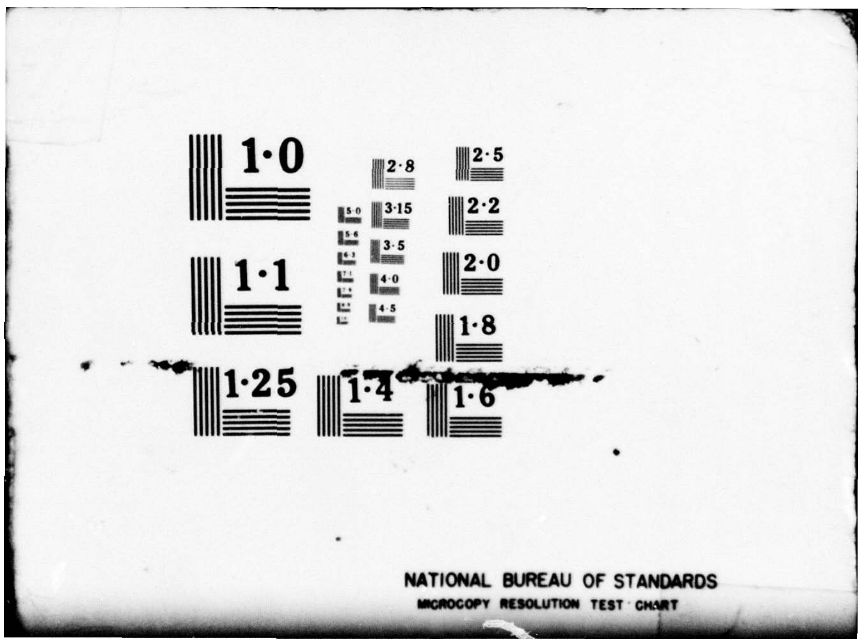
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# THE SEMICLASSICAL EXPANSION OF THE ANHARMONIC-OSCILLATOR PROPAGATOR

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# THE SEMICLASSICAL EXPANSION OF THE ANHARMONIC-OSCILLATOR PROPAGATOR[1]

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## ABSTRACT

This paper shows how to calculate the terms of a semiclassical (WKB) expansion of the quantum-mechanical propagator corresponding to the quartic anharmonic-oscillator potential, $V = m\omega^2 q^2/2 + \lambda q^4/4$. This nonperturbative treatment expresses each term in the series as a path integral, which is then evaluated in the framework of a formalism, introduced by C. DeWitt-Morette, which does not entail the usual time-slicing operation followed by a limiting procedure. The Gaussian measure used absorbs all the quadratic terms in the expansion of the action functional about a classical path. The covariance of this Gaussian measure is the Feynman Green function of the small-disturbance operator of the system. This function can be obtained by varying the constants of integration in the classical solution, and therefore the coefficients of the expansion depend only on this classical solution. If the latter is chosen to be the one which tends to its harmonic counterpart when $\lambda \rightarrow 0$, then it is seen that the propagator also tends to its harmonic counterpart when $\lambda \rightarrow 0$.

## I. INTRODUCTION

The one-dimensional quartic anharmonic oscillator is a particle of mass m in a potential given by:

$$V(q) = \frac{m\omega^2 q^2}{2} + \frac{\lambda q^4}{4} . \qquad (1)$$

It is an important model in physics as a prototype nonlinear field theory and has generated a great deal of activity in recent years for several reasons. First, it is a simple example of a perturbation which causes the associated quantum-mechanical quantities to be non-analytic in the coupling constant $\lambda$. Therefore, the usual perturbation series in powers of the coupling constant are divergent, although it has been shown[2] that the Padé approximants of the Rayleigh-Schrödinger series for the energy levels converge to the correct eigenvalues of the Hamiltonian, which has a positive-definite spectrum for $\lambda > 0$. The anharmonic oscillator is also the simplest nonlinear interaction which still yields plane-wave periodic solutions in the associated $\lambda\varphi^4$ field theory, and even admits of a restricted superposition principle[3].

While the energy spectrum has been studied rather extensively[2,4,5], the propagator $K \equiv \langle q_b, t_b | q_a, t_a \rangle$, or probability amplitude that a particle at $q_a$ at time $t_a$ will be at $q_b$ at time $t_b$, has not. The purpose of this paper is to show how to calculate the terms of a semiclassical (WKB) expansion of this propagator (in powers of $\hbar$). This treatment, of necessity nonperturbative since it does not hinge on any expansion in powers of $\lambda$, expresses each term in the series as a path integral. The latter is then evaluated in the framework of a formalism where the usual approach of time-slicing followed by a limiting procedure is replaced by a more tractable definition, introduced by C. DeWitt-Morette[6], which greatly simplifies calculations. This approach enabled us to systematically generate all the terms in the semiclassical expansion, which represents some progress over previous studies of approximating the anharmonic oscillator propagator by path-integral techniques (Lam[7], Sarkar[8], Mathews and Seshadri[9]).

First, the classical system is studied: the classical paths joining two fixed endpoints are calculated and the limit of zero coupling constant is discussed. Then, the classical action and other elements of the WKB expansion (Jacobi commutator, Van Vleck - Morette function, Feynman's Green function) are derived explicitly, and their connection with the small-disturbance equation investigated. Finally, the path integrals constituting the terms of the WKB expansion are exhibited and reduced to definite integrals over known functions, first for an arbitrary potential, then for the anharmonic oscillator.

## II. THE CLASSICAL SYSTEM

### The Potential

The potential, given in (1). is sketched below for $\lambda > 0$ and $\lambda < 0$.

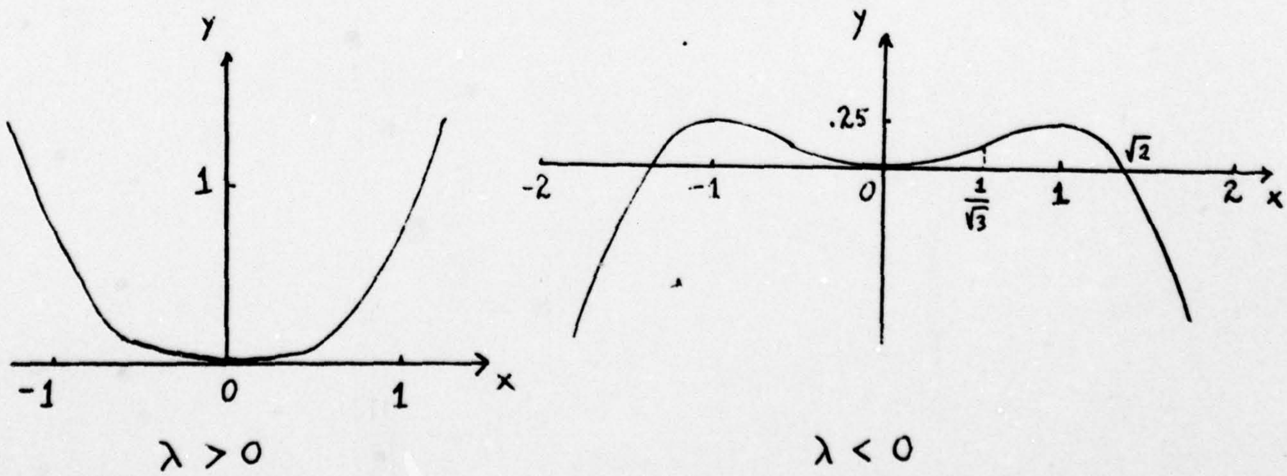


Figure 1. The anharmonic oscillator potential $V(q) = m\omega^2 q^2/2 + \lambda q^4/4$ [$y = x^2/2 \pm x^4/4$; $y \equiv V(q)|\lambda|/m^2\omega^4$, $x \equiv q|\lambda|^{1/2}/\omega m^{1/2}$].

The potential well is always present for $\lambda < 0$, so there will always be harmonic motion in some neighborhood of the origin. As $|\lambda|$ decreases, the well gets deeper and deeper, the maxima go higher and higher, and the points where the potential crosses the horizontal axis are rejected farther and farther. The drastic change in the shape of V as $\lambda$ changes sign is the cause for the nonanalyticity in $\lambda$. For $\lambda > 0$, there will always be a stable ground state, whereas for $\lambda < 0$, the ground state is unstable, as there is a finite probability for the particle to "leak out" of the well. The failure of perturbation theory is due to the fact that at large distances the $q^4$ term will always dominate the $q^2$ term, regardless how small $\lambda$ is.

Dynamical Equation

The dynamical equation for the classical path $q_c(t)$ is:

$$\ddot{q}_c(t) + \omega^2 q_c(t) + \frac{\lambda}{m} q_c^3(t) = 0 . \qquad (2)$$

It can be solved in terms of the (biperiodic) elliptic functions. Our source for the latter is Byrd and Friedman's handbook[10]. We choose the following form for the solution of (2):

$$q_c(t) = q_m \, \mathrm{cn}\left[\Omega(t - t_0), k\right], \qquad (3)$$

where

$$\Omega^2 \equiv \omega^2 + \lambda q_m^2 \quad , \quad k^2 \equiv \frac{\lambda}{2}\left(\frac{q_m}{\Omega}\right)^2 .$$

This corresponds to the case where the particle is released at $q_m$ at time $t=t_0$ with no initial velocity. (For simplicity, we take the mass m equal to 1; it can always be restored by replacing $\lambda$ by $\lambda/m$). Note that the modulus k lies always between 0 and $1/\sqrt{2}$ (= 0.707...). If we take the modulus k and the phase $t_0$ to be our constants of integration, we get:

$$q_c(t) = \sqrt{\frac{2k^2\omega^2}{\lambda(1-2k^2)}}\ \mathrm{cn}\left[\frac{\omega(t-t_0)}{\sqrt{1-2k^2}}, k\right]. \tag{4}$$

Classical paths

The classical paths of interest for the calculation of the propagator are those for which the initial and final positions are specified:

$$q_c(t_a) = q_a \qquad ; \qquad q_c(t_b) = q_b .$$

Substituting these conditions in (4) yields the relationship between the set $(k, t_0)$ and the set $(q_a, q_b)$:

$$\begin{aligned} &(a) \quad \frac{\omega(t_a - t_0)}{\sqrt{1-2k^2}} = \pm\, \mathrm{cn}^{-1}\left(\frac{q_a}{q_m}, k\right) + 4nK(k) \\ &(b) \quad \frac{\omega(t_b - t_0)}{\sqrt{1-2k^2}} = \pm\, \mathrm{cn}^{-1}\left(\frac{q_b}{q_m}, k\right) + 4n'K(k), \end{aligned} \tag{5}$$

where n and n' are integers,

$$q_m \equiv \sqrt{\frac{2k^2\omega^2}{\lambda(1-2k^2)}}, \tag{6}$$

and K(k) is the quarter-period of the cn function. Subtracting (5a) from (5b) yields the final transcendental equations giving k in terms of $q_a$ and $q_b$:

$$\pm \frac{\omega T}{\sqrt{1-2k^2}} = \varphi_{\pm}(k^2) + 4NK(k), \qquad (7)$$

where any combination of signs is permitted, $T \equiv t_b - t_a$, N is an integer, and

$$\varphi_{\pm}(k^2) \equiv cn^{-1}\left(\frac{q_a}{q_m}\right) \pm cn^{-1}\left(\frac{q_b}{q_m}\right).$$

Equation (7) must be solved graphically for k ($t_0$ is then determined, for example, by Equation 5a). Since $cn^{-1}u$ is defined only for $u \in [-1,1]$, we must have $|q_a| \leq q_m$ and $|q_b| \leq q_m$. Thus, in addition to the upper cutoff ½ on $k^2$, we have a lower cutoff:

$$k^2_{min} \leq k^2 \leq ½,$$

where

$$k^2_{min} \equiv \frac{\lambda}{2(\lambda + \omega^2\alpha^2)} \ ; \qquad \alpha \equiv \frac{1}{max(|q_a|, |q_b|)} .$$

Note that $cn^{-1}$ is always positive. It monotonically decreases from $cn^{-1}(-1) = 2K(k)$ to $cn^{-1}(1) = 0$, with an inflexion point at (0, K(k)).

A sample graphical solution of (7) is shown in Figure 2, for $\omega = T = q_a = q_b = 1$. The cases $\lambda$ = 0.001, 0.5, and 1 are shown. The curve $\omega T/\sqrt{1-2k^2}$ intersects $\varphi_+(k^2)$ once, twice, or not at all. Each intersection gives the modulus k for a possible classical path such that $q(t_a) = q_a$ and $q(t_b) = q_b$. There comes a point where each of the curves $\varphi_+(k^2) + 4NK(k)$ (one for each N) intersects $\omega T/\sqrt{1-2k^2}$ twice for each $N > N_0$. Therefore, there is always a countably infinite number of paths, with a cluster point at $k^2$ = ½.

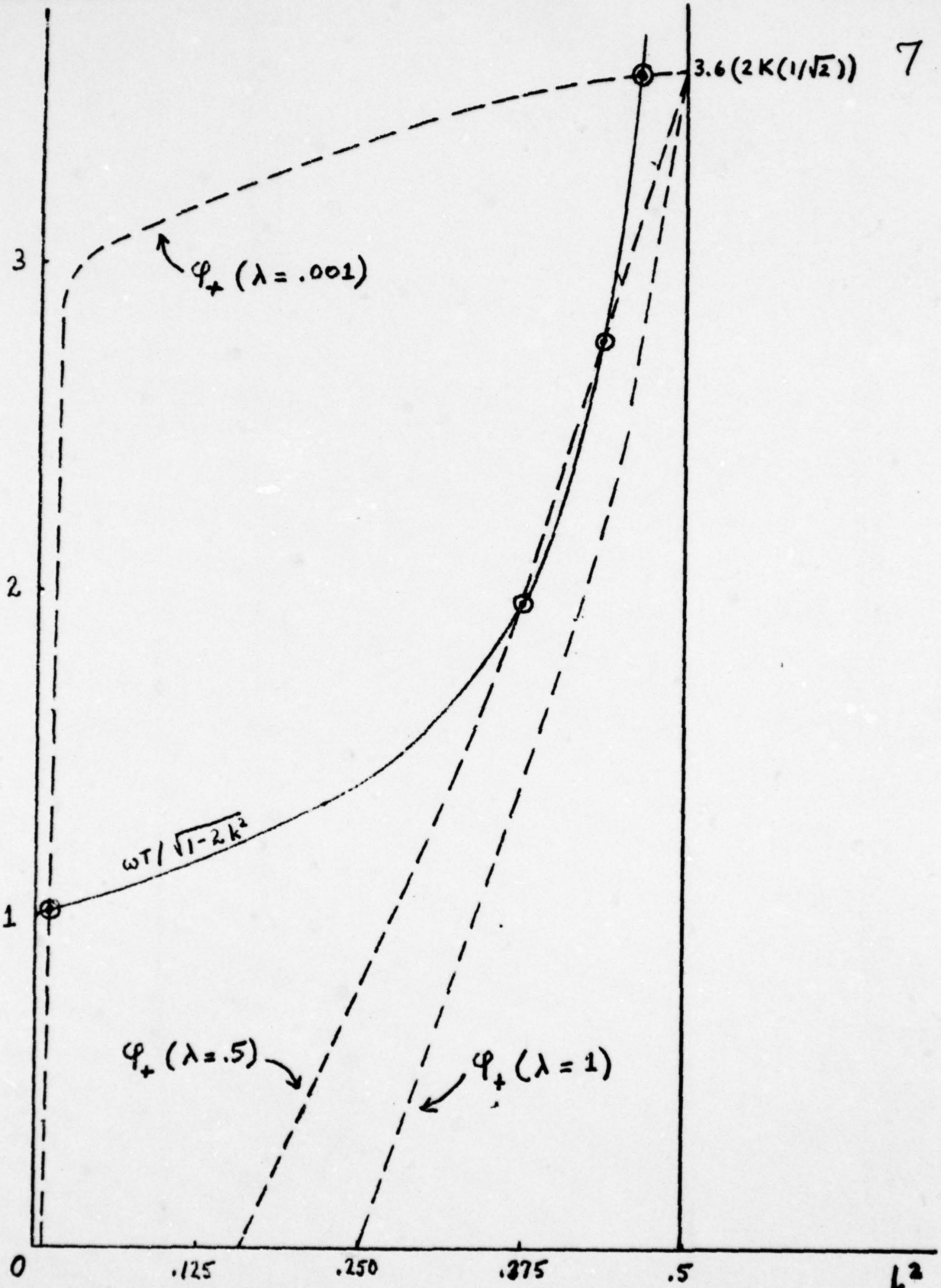


Figure 2. Classical paths for the anharmonic oscillator. Each intersection (circled) gives a value of k which corresponds to a classical solution of the dynamical equation for fixed-endpoint boundary conditions.

The higher the k, the higher the amplitude of the corresponding path (as revealed by Equation 4).

<u>Behavior as $\lambda \rightarrow 0$.</u> We shall be particularly concerned with the behavior of our expressions as $\lambda$ approaches 0. What happens to the classical solution as $\lambda \rightarrow 0$? For initial boundary conditions, it appears, according to (3), that we retrieve harmonic motion: indeed, as $\lambda \rightarrow 0$, $k \rightarrow 0$, $\Omega \rightarrow \omega$, and $cn \rightarrow \cos$. However, for other boundary conditions, it appears, according to (4), that we have a $1/\sqrt{\lambda}$ singularity as $\lambda \rightarrow 0$: indeed, for arbitrary values of the constants of integration (say $k^2 = 0.3$ and $t_0 = 2$ seconds), (4) indicates that $q_c(t) \sim 1/\sqrt{\lambda}$ as $\lambda \rightarrow 0$. Is harmonic motion irretrievable then as a limiting case?

The answer is no. The reason is that only <u>physical</u> boundary conditions (such as position and velocity at certain times) are acceptable[11]. $k^2 = 0.3$ is <u>not</u> a physical boundary condition. When the latter are inserted, k <u>will depend on $\lambda$</u> in such a manner as to make at least one classical path $q_c(t)$ reduce to harmonic motion when $\lambda \rightarrow 0$.

In the case of endpoint boundary conditions, (4) shows that the only way that $q_c(t)$ can retain its constant, preassigned values at $t_a$ and $t_b$ is if $k^2$ goes to 0 as fast as $\lambda$. The ratio $k^2/\lambda$ is then an arbitrary constant A, which may be dependent on $\omega$, and (4) becomes $q_c(t) = A\cos\omega(t-t_0)$, which is harmonic motion. Figure 2 shows that as $\lambda$ approaches 0 there is always one solution $k^2$ which also approaches 0. This solution, which we call $q_{co}(t)$, is the lowest-amplitude (or lowest-energy) path, and coincides, when $\lambda = 0$, with the (generally) unique harmonic-oscillator path between the two fixed endpoints. The other paths correspond to values of k which do not go to 0 with $\lambda$, and hence their amplitudes increase without bound as $\lambda \rightarrow 0$. Their graph becomes, in the limit, a set of parallel lines perpendicular to the t-axis, one of which going through $t_a$ and the other through $t_b$.

Our semiclassical expansion of the propagator will be about this regular path $q_{co}(t)$. Since all the coefficients will depend, directly or indirectly, on $q_{co}$ alone, the anharmonic propagator will tend toward the harmonic propagator as the coupling constant tends to 0.

Classical action

The classical action (or action functional evaluated at a classical path) for the anharmonic oscillator is needed for the WKB approximation. It is given by:

$$S_c \equiv \int_{t_a}^{t_b} L\left(q_c(t), \dot{q}_c(t), t\right) dt$$
$$= \int_{t_a}^{t_b} \left[\frac{1}{2}\dot{q}_c^2(t) - \frac{1}{2}\omega^2 q_c^2(t) - \frac{1}{4}\lambda q_c^4(t)\right] dt .$$

Using the integrals 312.02 (p. 193), 361.02 (p. 212), 312.04 (p. 193) of Reference 10, and the formula

$$E(u') - E(u) = E(u'-u) - k^2 \operatorname{sn} u . \operatorname{sn} u' . \operatorname{sn}(u'-u) , \tag{8}$$

[derived from formulas 116.01 (p. 13) and 123.01 (p. 23) of Reference 10], we obtain the answer:

$$S_c = \frac{-2\omega^3}{3\lambda\sqrt{1-2k^2}} E\left(\frac{\omega T}{\sqrt{1-2k^2}}\right) + \frac{2\omega^3 k^2}{3\lambda\sqrt{1-2k^2}} \Big[ sn(u_a)\, sn(u_b) \cdot$$

$$\cdot\, sn \frac{\omega T}{\sqrt{1-2k^2}} + \frac{1}{1-2k^2}\Big( sn(u_a)\, cn(u_a)\, dn(u_a) - \qquad (9)$$

$$- sn(u_b)\, cn(u_b)\, dn(u_b)\Big)\Big] + \frac{\omega^4(1-k^2)(2-3k^2)\,T}{3\lambda(1-2k^2)^2},$$

where $u_{a/b} \equiv \omega(t_{a/b} - t_o)/\sqrt{1-2k^2}$.

Behavior when $\lambda \to 0$. Let us look at the behavior of $S_c$ as $\lambda \to 0$ along the path $q_{co}$, where $k^2 \to 0$ as $\lambda \to 0$ such that $k^2/\lambda$ is a constant. Using the fact that $E(u) = u + O(k^2)$, we easily see that $S_c$ is regular at $\lambda = 0$, and reduces to the classical action for the harmonic oscillator.

## III. THE QUANTUM SYSTEM

### The Small-Disturbance Equation

Just as the classical system is dominated by the dynamical (or Euler-Lagrange) equation, the quantum system is dominated by the small-disturbance (or Jacobi) equation. The latter is the equation satisfied by a small variation in the classical path, obtained, for example, by a small change in a constant of integration, such as the total energy or an endpoint. The small-disturbance equation is studied in more detail in Appendix A and in References 1 and 6c. For the anharmonic oscillator, it is:

$$\left[-\frac{d^2}{dt^2} - \omega^2 - 3\lambda q_c^2(t)\right] f(t) = 0. \qquad (10)$$

### Solutions of the small-disturbance equation

Solutions of the small-disturbance equation can always be generated by differentiating the classical solution with respect to a constant of integration.

This simple procedure was known to Jacobi[12], but it seems to be sometimes forgotten today, as one still finds attempts at solving the equation directly; for example, Sarkar[8] has undertaken this very difficult task for the anharmonic oscillator (Equation 10).

The functions we will need for the path-integral treatment of the propagator are the Jacobi commutator $J(t,t')$, the Van Vleck - Morette (VVM) function $M(t_a,t_b)$, and the Feynman Green function $G(t,t')$. Their expressions are given below, followed by their definition and derivation.

Jacobi commutator

$$
\begin{aligned}
J(t,t') = \frac{(1-2k^2)^{3/2}}{\omega} \operatorname{sn} u \operatorname{sn} u' \operatorname{dn} u \operatorname{dn} u' \cdot \\
\Bigg[ \frac{-1}{1-2k^2} \left( \frac{\operatorname{cn} u'}{\operatorname{sn} u' \operatorname{dn} u'} - \frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u} \right) + \frac{u'-u}{1-2k^2} \\
- \frac{E(u'-u)}{k'^2} + \frac{k^2}{k'^2} \Bigg( \frac{\operatorname{sn} u' \operatorname{cn} u'}{\operatorname{dn} u'} - \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} \\
+ \operatorname{sn} u \operatorname{sn} u' \operatorname{sn}(u'-u) \Bigg) \Bigg],
\end{aligned}
\tag{11}
$$

where

$$
u \equiv \frac{\omega(t-t_0)}{\sqrt{1-2k^2}}, \quad u' \equiv \frac{\omega(t'-t_0)}{\sqrt{1-2k^2}}, \quad \text{and} \quad k' = 1-k^2;
$$

VVM function

$$
M(t_a,t_b) = [J(t_a,t_b)]^{-1}; \tag{12}
$$

Feynman's Green function

$$
G(t,t') = \frac{J(t',t_a)J(t_b,t)Y(t-t') + J(t,t_a)J(t_b,t')Y(t'-t)}{J(t_a,t_b)}. \tag{13}
$$

Definitions and derivations

The Jacobi commutator. This function J(t,t') of two variables can be defined as follows: the unique, retarded Green function of the small-disturbance operator, satisfying

$$\left[-\frac{d^2}{dt^2} - \omega^2 - 3\lambda q_c^2(t)\right] G(t,t') = \delta(t-t'), \qquad (14)$$

is $G^-(t,t') = J(t,t')Y(t-t')$, where $Y(x) \equiv 1$ for $x>0$ and 0 otherwise. J(t,t') is antisymmetric and satisfies the small-disturbance equation in both t and t'. It is called the commutator because, as shown in Appendix A, it can be written as a Poisson bracket of position at different times with respect to initial (or final) position and momentum; when the system is quantized, this expression becomes the commutator. For example, for initial boundary conditions, we have:

$$J(t,t') = \frac{\partial q_c(t)}{\partial q_a}\frac{\partial q_c(t')}{\partial p_a} - \frac{\partial q_c(t')}{\partial q_a}\frac{\partial q_c(t)}{\partial p_a} \qquad (15)$$

$$\equiv \left\{ q_c(t), q_c(t') \right\}_{(q_a, p_a)} \rightarrow \frac{1}{i\hbar}\left[\underset{\sim}{Q}(t), \underset{\sim}{Q}(t')\right].$$

For any two convenient constants of integration $\alpha_1$ and $\alpha_2$ we can write the commutator as (see proof in Appendix A):

$$J(t,t') = \frac{\dfrac{\partial q_c(t)}{\partial \alpha_1}\dfrac{\partial q_c(t')}{\partial \alpha_2} - \dfrac{\partial q_c(t')}{\partial \alpha_1}\dfrac{\partial q_c(t)}{\partial \alpha_2}}{\dfrac{\partial q_c(t_b)}{\partial \alpha_1}\dfrac{\partial p_c(t_b)}{\partial \alpha_2} - \dfrac{\partial p_c(t_b)}{\partial \alpha_1}\dfrac{\partial q_c(t_b)}{\partial \alpha_2}} \qquad (16)$$

(or a similar expression with $t_b$ replaced by $t_a$), where $p_c(t)$ is the classical momentum (equal to $\dot{q}_c(t)$ for the anharmonic oscillator). We will use this formula with $q_c(t)$ given by (4) and $\alpha_1 = k$, $\alpha_2 = t_0$. The velocity is given by:

$$\dot{q}_c(t) = \frac{-k\omega^2}{1-2k^2}\sqrt{\frac{2}{\lambda}}\ \mathrm{sn}\,u.\mathrm{dn}\,u = -\frac{\partial q_c(t)}{\partial t_0},$$

where u is defined in (11). The formulas for differentiating the elliptic functions with respect to the modulus k are found in Reference 10 (710.51-3, p. 283). Since the argument of the elliptic functions also depends on k, the chain rule must be used to evaluate $\partial q_c(t)/\partial k$ and $\partial \dot{q}_c(t_b)/\partial k$. We obtain:

$$\frac{\partial q_c(t)}{\partial k} = \omega\sqrt{\frac{2}{\lambda}}\frac{\mathrm{cn}\,u}{(1-2k^2)^{3/2}} - \sqrt{\frac{2k^2\omega^2}{\lambda(1-2k^2)}}\ \mathrm{sn}\,u.\mathrm{dn}\,u\left\{\frac{2k\omega(t-t_0)}{(1-2k^2)^{3/2}}\right.$$
$$\left. + (1/kk'^2)\left[-E(u) + k'^2u + k^2\,\mathrm{sn}\,u.\mathrm{cn}\,u/\mathrm{dn}\,u\right]\right\}.$$

The denominator in (16) is calculated to be:

$$\frac{\partial q_b}{\partial k}\frac{\partial \dot{q}_c(t_b)}{\partial t_0} - \frac{\partial \dot{q}_c(t_b)}{\partial k}\frac{\partial q_b}{\partial t_0} = \frac{2\omega^4 k}{\lambda(1-2k^2)^3}.$$

These formulas, along with (8), lead us to the stated expression (11) for J(t,t'). We see that for $q_c = q_{co}$, i.e. when $k^2$ goes to 0 with $\lambda$, we have $E(u)\rightarrow u$, $\mathrm{dn}\,u \rightarrow 1$, $\mathrm{sn}\,u \rightarrow \sin u$, $\mathrm{cn}\,u \rightarrow \cos u$, $u \rightarrow \omega(t-t_0)$, and $J(t,t') \rightarrow \omega^{-1}\sin\omega(t'-t)$, which is the harmonic-oscillator commutator function.

<u>The VVM function</u>. The WKB approximation to the propagator is given by the well-known formula:

$$K_{WKB} = (M/2\pi i\hbar)^{1/2} \exp(iS_c/\hbar), \tag{17}$$

where

$$M \equiv -\frac{\partial^2 S_c}{\partial q_a \partial q_b} = -\frac{\partial \dot{q}_c(t_b)}{\partial q_a} \tag{18}$$

is the Van Vleck - Morette function. The second expression for M, which will be used in the evaluation, uses the fact that $\partial S_c/\partial q_b = p_c(t_b) = \dot{q}_c(t_b)$.[13] Therefore, to get M in terms of k and $t_0$ we must use the chain rule:

$$M = -\frac{\partial \dot{q}_c(t_b)}{\partial k}\frac{\partial k}{\partial q_a} - \frac{\partial \dot{q}_c(t_b)}{\partial t_0}\frac{\partial t_0}{\partial q_a} . \tag{19}$$

In order to calculate M, we must express $\partial k/\partial q_a$ and $\partial t_0/\partial q_a$ in terms of $\partial q_a/\partial k$, $\partial q_a/\partial t_0$, etc. Since we must have

$$\begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$k_1 \equiv \frac{\partial q_a}{\partial k} \qquad k_2 \equiv \frac{\partial q_a}{\partial t_0} \qquad u_1 \equiv \frac{\partial k}{\partial q_a} \qquad u_2 \equiv \frac{\partial k}{\partial q_b}$$

$$k_3 \equiv \frac{\partial q_b}{\partial k} \qquad k_4 \equiv \frac{\partial q_b}{\partial t_0} \qquad u_3 \equiv \frac{\partial t_0}{\partial q_a} \qquad u_4 \equiv \frac{\partial t_0}{\partial q_b},$$

we can easily solve for the u's in terms of the k's, to get

$$u_1 = \frac{k_4}{\mathcal{D}}, \quad u_2 = -\frac{k_2}{\mathcal{D}}, \quad u_3 = -\frac{k_3}{\mathcal{D}}, \quad u_4 = \frac{k_1}{\mathcal{D}},$$

where $\mathcal{D} \equiv k_1k_4 - k_2k_3$. Substituting this result in (19), and comparing with (16), we see that we get the value of M stated in (12), namely $M = [J(t_a,t_b)]^{-1}$.

Feynman's Green function. Feynman's Green function $G(t,t')$, satisfying (14), is the unique Green function of the small-disturbance operator which vanishes at both endpoints. It is important for our treatment because it is the covariance of the Gaussian measure used to express the propagator as a path integral. As was stated before (and proved in Appendix A), $G^-(t,t') \equiv J(t,t')Y(t-t')$, with J as in (16), satisfies (14). The function

$$G(t,t') \equiv J(t,t')Y(t-t') + \frac{J(t,t_a)J(t_b,t')}{J(t_a,t_b)}$$

is also a Green function, since the addition to $G^-(t,t')$ is a homogeneous solution of the small-disturbance equation in t and t'. Further, $G(t_a,t') = G(t_b,t') = 0$. Therefore, $G(t,t')$ is Feynman's Green function. To put it in the form given in (13) requires use of the identity

$$J(t,t') = \frac{J(t',t_a)J(t_b,t) - J(t,t_a)J(t_b,t')}{J(t_a,t_b)},$$

easily proved by using (16).

## IV. WKB EXPANSION OF THE PROPAGATOR BY PATH INTEGRALS

### Arbitrary Potential

The framework for a WKB expansion of the propagator by phase-space path integrals without limiting procedure was set in an earlier paper[14] and will be only briefly summarized here. For a simple Hamiltonian of the form $p^2/2m + V(q,t)$ considered here, the phase-space path integral becomes a configuration-space path integral, since the momentum-dependent terms are rolled into the measure and only position-dependent terms remain to be path-integrated. The first step is to expand the classical action functional about the classical path $q_c(t)$:

$$S[q] \equiv S[q_c + x] = S_c + \frac{1}{2}\int_T \left[\dot{x}^2(t) - V''(t)x^2(t)\right]dt$$
$$- \sum_{n=3}^{\infty} \frac{1}{n!}\int_T V^{(n)}(t)\,x^n(t)\,dt,$$

where

$$V^{(n)}(t) \equiv \left[\partial^n V(q,t)/\partial q^n\right]_{q=q_c},$$

$T \equiv [t_a, t_b]$, and $x \in \mathcal{C}_0$, the space of paths such that $x(t_a) = x(t_b) = 0$. The classical action $S_c$ becomes part of the WKB approximation, $K_{WKB}$, and the quadratic terms are rolled into the Gaussian measure, leaving the sum term for path integration. The result is:

$$K = K_{WKB}\int_{\mathcal{C}_0} dw_0(x)\exp\left\{\frac{-i}{\hbar}\sum_{n=3}^{\infty}\int_T V^{(n)}(t)\,\frac{x^n(t)}{n!}\,dt\right\}, \tag{20}$$

where the measure $w_0$ is defined by its Fourier transform:

$$\mathcal{F}w_0(\mu) \equiv \exp\left[-\frac{i\hbar}{2}\iint_{T\,T} G(t,t')\,d\mu(t)\,d\mu(t')\right],$$

$G(t,t')$ being Feynman's Green function defined earlier and $\mu$ being a bounded measure on the time-interval T. $K_{WKB}$ is given by (17). The exponential in (20) can be expanded to yield:

$$K = K_{WKB}\Big[1 + \sum_{j=1}^{\infty} \frac{1}{j!}\left(\frac{-i}{\hbar}\right)^{j} \sum_{n_1=3}^{\infty} \cdots \sum_{n_j=3}^{\infty} \int_{T^j} \frac{dt_1 \ldots dt_j}{n_1! \ldots n_j!} \tag{21}$$

$$\times V^{(n_1)}(t_1)\ldots V^{(n_j)}(t_j) \int_{\mathcal{C}_0} x^{n_1}(t_1)\ldots x^{n_j}(t_j)\, dw_0(x)\Big].$$

To evaluate the path integral, we need the moments formula [see, e.g., Reference 15]:

$$\int_{\mathcal{C}_0} x(t_1)x(t_2)\ldots x(t_n)dw_0(x) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (i\hbar)^m \sum' G(t_{i_1},t_{i_2})G(t_{i_3},t_{i_4}) \times \\ \quad \ldots \times G(t_{i_{2m-1}},t_{i_{2m}}) & \text{if } n=2m \text{ is even,} \end{cases} \tag{22}$$

where $\sum'$ denotes the sum over all different combinations of different indices $i_i$, with $\{i_1,i_2,\ldots,i_n\} \equiv \{1,2,\ldots,n\}$. There are $(2m-1)!! \equiv (2m-1)(2m-3)\ldots 5.3.1$ terms in all for $n = 2m$.[16]

Thus, we see that $\hbar$ comes in the expansion with power $\frac{1}{2}(n_1+\ldots+n_j) - j$, which is always a positive integer, since each $n_i$ is at least 3. This proves that (21) is indeed an expansion in powers of $\hbar$, and we can write:

$$K = K_{WKB}\left(1 + \hbar K_1 + \hbar^2 K_2 + \ldots\right), \tag{23}$$

where the $K_i$s are ordinary finite-dimensional integrals over the time-interval T. Polynomial potentials are best suited for this scheme, since the expansion of the action terminates at some finite n. However, it is important to note that regardless of the potential each term in the WKB expansion (coefficient of $\hbar^k$) is <u>always</u> a terminating series. For example, inspection of (21) shows that the first (post-WKB) term is, for arbitrary potential:

$$\hbar K_1 = \left(-\frac{i}{\hbar}\right)\int_T \frac{dt}{4!} V^{(4)}(t) \int_{\mathcal{C}_0} x^4(t)\, dw_0(x) \tag{24}$$

$$+ \frac{1}{2}\left(-\frac{i}{\hbar}\right)^2 \int_{T^2} \frac{dt}{3!}\frac{ds}{3!} V^{(3)}(t) V^{(3)}(s) \int_{\mathcal{C}_0} x^3(t) x^3(s)\, dw_0(x),$$

and the moments formula gives:

$$K_1 = \frac{-i}{8} \int_T V^{(4)}(t) G^2(t,t)\, dt \tag{25}$$

$$+ \frac{i}{24} \int_{T^2} V^{(3)}(t) V^{(3)}(s) \left[3G(t,t)G(t,s)G(s,s) + 2G^3(t,s)\right] dt\, ds.$$

Let us study the structure of the coefficients $K_i$. In general, the $j = 1$ term in (19) is:

$$\left(-\frac{i}{\hbar}\right) \sum_{n=3}^{\infty} \int_T \frac{dt}{n!} V^{(n)}(t) \int_{\mathcal{C}_0} x^n(t)\, dw_0(x) \tag{26}$$

$$= \sum_{m=2}^{\infty} \frac{(i\hbar)^{m-1}}{m!\, 2^m} \int_T V^{(2m)}(t)\, G^m(t,t)\, dt.$$

For arbitrary potentials, this is an infinite series in $\hbar$ with no constant term. Similarly, we find that:

a) In the series for $j = 2$, the $n_1 = n_2 = 3$ term is the term proportional to $\hbar$, and the three terms $n_1 = n_2 = 4$; $n_1 = 3$, $n_2 = 5$; and $n_1 = 5$, $n_2 = 3$ are the one proportional to $\hbar^2$. All the subsequent j series start out with $\hbar^k$ for $k \geq 2$.

b) In the series for $j = 3$, the three terms $n_1 = n_2 = 3$, $n_3 = 4$; $n_1 = 4$, $n_2 = n_3 = 3$; and $n_1 = n_3 = 3$, $n_2 = 4$ are the only ones proportional to $\hbar^2$, and the $n_1 = n_2 = n_3 = 4$ term is the only one proportional to $\hbar^3$.

c) In the series for $j = 4$, the term $n_1 = n_2 = n_3 = n_4 = 3$ is the only one proportional to $\hbar^2$.

d) The series for $j = 5$ starts out with the $\hbar^3$ term.

Thus, we can write the term proportional to $\hbar^2$ in the expansion:

$$\hbar^2 K_2 = \frac{-\hbar^2}{48}\int_T V^{(6)}(t)\, G^3(t,t)\,dt$$

$$-\frac{1}{2\hbar^2}\int_{T^2}\frac{dt_1}{4!}\frac{dt_2}{4!}V^{(4)}(t_1)V^{(4)}(t_2)\int_{\mathcal{C}_0} x^4(t_1)x^4(t_2)\cdot d w_0(x)$$

$$-\frac{1}{2\hbar^2}\int_{T^2}\frac{dt_1}{3!}\frac{dt_2}{5!}V^{(3)}(t_1)V^{(5)}(t_2)\int_{\mathcal{C}_0} x^3(t_1)x^5(t_2)\,dw_0(x)$$

$$+\frac{i}{6\hbar^3}\int_{T^3}\frac{dt_1}{3!}\frac{dt_2}{3!}\frac{dt_3}{4!}V^{(3)}(t_1)V^{(3)}(t_2)V^{(4)}(t_3)\cdot \int_{\mathcal{C}_0} x^3(t_1)\,x^3(t_2)x^4(t_3)\,dw_0(x)$$

$$+\frac{1}{4!}\frac{1}{\hbar^4}\int_{T^4}\frac{dt_1\cdots dt_4}{(3!)^4}V^{(3)}(t_1)V^{(3)}(t_2)V^{(3)}(t_3)V^{(3)}(t_4)\cdot \int_{\mathcal{C}_0} x^3(t_1)\,x^3(t_2)x^3(t_3)\,x^3(t_4)\,dw_0(x), \tag{27}$$

and the moments formula (22) gives the value of the path integrals in terms of Feynman's Green function and the classical path.

Application to the Anharmonic Potential

The anharmonic oscillator potential, given by (1), is $V(q) = m\omega^2 q^2/2 + \lambda q^4/4$. The first-order correction to the WKB approximation is then given by (25):

$$K_1 = \frac{-3\lambda i}{4} \int_T G^2(t,t)\, dt + \frac{3\lambda^2 i}{2} \int_{T^2} dt\, ds\, q_c(t) q_c(s) \cdot$$

$$\times \left[ 3\, G(t,t) G(t,s) G(s,s) + 2\, G^3(t,s) \right], \tag{28}$$

where $G(t,t')$ is given explicitly by (13) with J given by (11) and $q_c(t)$ by (4). The resulting integrals over the elliptic functions are all well-known and of the type tabulated in Reference 10. Higher-order corrections can be generated at will, although they generally involve a large number of integrals. The WKB approximation is given by (17), with the classical action $S_c$ given by (9) and the VVM function M given by (12), with J in (11). Therefore, every function entering the semiclassical expansion of the anharmonic oscillator propagator has been explicitly calculated, and the definite integrals giving the coefficients of the expansion have been explicitly exhibited. It is pointed out, again, that this treatment is nonperturbative, since the functions involved in the terms of the expansion, for example $q_c$ and G in (28), depend implicitly on $\lambda$ . This example illustrates the power of path integration without limiting procedure.

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# FOOTNOTES

1. This paper is based in part on the author's Ph.D. dissertation, "An Investigation of the Feynman Path Integral Formulation of Quantum Mechanics", The University of Texas at Austin, Austin, Texas, August 1975.

2. J. J. Loeffel, A. Martin, B. Simon, and A. S. Wightman, Phys. Lett. 30B(1969), 656-8, and Barry Simon, Ann. of Phys. 58 (1970), 76-136.

3. Indeed, the dynamical equation for the $\lambda\varphi^4$ self-interaction, $(\Box - m^2)\varphi - \lambda\varphi^3 = 0$, can be readily reduced to the dynamical equation for the one-dimensional anharmonic oscillator, namely $\bar{\varphi}'' - m^2\bar{\varphi}/K^2 - \lambda\bar{\varphi}^3/K^2 = 0$, where $\varphi(x_1,x_2,x_3,x_4) \equiv \bar{\varphi}(K.x)$, K being an arbitrary four-vector (plane-wave solution). The elliptic functions which are solutions of this equation are periodic, and admit of a restricted superposition principle, rare for non-linear equations: if an elliptic cosine (cn) with a certain modulus $k_1$ is a solution, and if an elliptic sine (sn) with another modulus $k_2$ is also a solution, then the linear combination (cn + i sn) is also a solution, but the common modulus $k_3$ is different from $k_1$ and $k_2$ (cf. Gérard Pétiau, Cahiers de Physique 14 (1960), 1-24, and D. F. Kurdgelaidzé, Cahiers de Physique, No. 128 (1961), 149-57).

4. Carl M. Bender and Tai Tsun Wu, Phys. Rev. Lett. 21 (1968), 406-9; Phys. Rev. 184 (1969), 1231-60; Phys. Rev. Lett. 27 (1971), 461-5; Phys. Rev. D7 (1973), 1620-36.

5. P. M. Mathews and K. Eswaran, Lett. Nuov. Cim. 5 (1972), 15-8.

6. C. DeWitt-Morette, (a) Comm. Math. Phys. 28 (1972), 47-67; (b) 37 (1974), 63-81; (c) Ann. of Phys. 97 (1976), 367-99; (d) (with A. Maheshwari and B. Nelson), to appear in Phys. Rep.

7. C. S. Lam, Nuov. Cim. Serie X, 47 (1967), 451-69; 50 (1967), 504-10.

8. S. Sarkar, Phys. Rev. D8 (1973), 1060-7.

9. P.M. Mathews and M. S. Seshadri, Int. Jour. of Theor. Phys. 13 (1975), 279-88.

10. Paul F. Byrd and Morris D. Friedman, Handbook of Elliptic Integrals for Engineers and Physicists. Berlin: Springer-Verlag, 1954.

11. We can better understand this question of physical boundary conditions from the simpler example of a particle in free fall with friction taken into account. The dynamical equation is $\ddot{x} = -(g + k\dot{x})$, with solution $x(t) = -gt/k + Ak^{-2}e^{-kt} + B$, where A and B are constants of integration. When $k \rightarrow 0$, we expect to retrieve free fall: $x(t) = -gt^2/2 + v_0 t + x_0$. Instead, we find a "singularity" at $k = 0$ if A and B are numerically specified. However, numerical specification of A and B does not constitute physical boundary conditions. Physical boundary conditions, such as $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$, always make A and B depend on k in such a manner as to make the solution, [namely, in this case, $x(t) = x_0 - gt/k + (g + v_0 k)(1 - e^{-kt})/k^2$], reduce properly when $k \rightarrow 0$.

12. Jacobi, "On the theory of the calculus of variations and of differential equations", Crelle's Mathematical Journal 17 (1837), referred to in Bolza's Calculus of Variations, p. 56.

13. This relation and similar ones can be simply derived as follows. For any Lagrangian L in n dimensions and $u \equiv t_a, t_b, q_a$, or $q_b$, we have $\partial S_c/\partial u$

$$= (\partial/\partial u)\int_{t_a}^{t_b} L(q_c, \dot{q}_c, t)\,dt = (p_c)_i(t_b)[\partial q_c^i(t)/\partial u]_{t=t_b} - (p_c)_i(t_a) \times [\partial q_c^i(t)/\partial u]_{t=t_a} + \epsilon L(u),$$

where $\epsilon = 0$ for $u = q_a$ or $u = q_b$, 1 for $u = t_b$ and -1 for $u = t_a$, since $(p_c)_i(t) \equiv [\partial L/\partial \dot{q}^i]_{q=q_c}$. Given that the classical Hamiltonian is $H_c(p_c, q_c, t) \equiv (p_c)_i(\dot{q}_c)^i - L(q_c, \dot{q}_c, t)$, this gives the following 4 relations of Hamilton-Jacobi theory:

$$\partial S_c/\partial q_a^i = -(p_c)_i(t_a) \qquad \partial S_c/\partial t_a = H_c(-\partial S_c/\partial q_a, q_a, t_a)$$

$$\partial S_c/\partial q_b^i = (p_c)_i(t_b) \qquad \partial S_c/\partial t_b = -H_c(\partial S_c/\partial q_b, q_b, t_b).$$

14. Maurice M. Mizrahi, J. Math. Phys. 19 (1978), 298-307.

15. Maurice M. Mizrahi, J. Math. Phys. 17 (1976), 566-75.

16. Among the specific moments needed are the following:

$$\text{a)} \int_{\mathcal{C}_0} x^{2n}(t)\,dw_0(x) = (2n)!\, \hbar^n\, i^n\, G^n(t,t)/2^n n!$$

$$\text{b)} \int_{\mathcal{C}_0} x^2(t)x^2(t')\,dw_0(x) = -\hbar^2 [G(t,t)G(t',t') + 2G^2(t,t')]$$

$$\text{c)} \int_{\mathcal{C}_0} x^3(t)x^3(t')dw_0(x) = -i\hbar^3 [9G(t,t)G(t,t')G(t',t') + 6G^3(t,t')]$$

$$\text{d)} \int_{\mathcal{C}_0} x^2(t)x^4(t')dw_0(x) = -i\hbar^3 [12G^2(t,t')G(t',t') + 3G(t,t)G^2(t',t')]$$

$$\text{e)} \int_{\mathcal{C}_0} x^4(t)x^4(t')dw_0(x) = \hbar^4 [9G^2(t,t)G^2(t',t') + 24G^4(t,t') + 72\, G(t,t)G(t',t')G^2(t,t').$$

For higher moments, we need a compressed notation. We write:

$$12\, \hbar^3\, G^2(t_1,t_2)\, G(t_2,t_2) \equiv 12\, (12)^2\, (22).$$

Then,

$$\begin{aligned} \text{f)} \int_{\mathcal{C}_0} x^3(t_1)x^3(t_2)x^4(t_3)dw_0(x) &= 27\, (11)\, (12)\, (22)\, (33)^2 \\ &+ 18\, (12)^3\, (33)^2 + 72\, (31)^3\, (32)\, (22) \\ &+ 72\, (31)\, (32)^3\, (11) + 108\, (33)\, (31)\, (32)\, (11)\, (22) \\ &+ 108\, (33)\, (32)^2\, (11)\, (12) + 108\, (33)\, (31)^2\, (22)\, (21) \\ &+ 216\, (31)^2\, (32)^2\, (12) + 216\, (33)\, (31)\, (32)\, (12)^2. \end{aligned}$$

There are $(3+3+4 - 1)!! = 9 \times 7 \times 5 \times 3 = 945$ terms in all .

17. J. Milnor, Morse Theory, Based on lecture notes by M. Spivak and R. Wells. Princeton University Press, 1969. Annals of Mathematics Studies, No. 51.

18. Bryce S. DeWitt, Dynamical Theory of Groups and Fields. New York: Gordon and Breach, 1965.

19. C. DeWitt-Morette, in Long-Term Predictions in Dynamics, edited by V. Szebehely and B. D. Tapley, Dordrecht: D. Reidel, 1976, pp. 57-65; also pp. 67-70 (with Pete Tschumi).

# APPENDIX A -- THE SMALL-DISTURBANCE EQUATION

This appendix will derive and generalize some results used in the text on the equation of small disturbances. The latter, resulting from the second variation of the action functional, is satisfied by the variation in a classical path resulting from a small change in the boundary conditions. For example, let $S[q] = S[\bar{\beta}(u)]$ be an action functional. Each path $q \equiv \bar{\beta}(u)$ is characterized by a parameter u: $q(t) = \bar{\beta}(u)(t) \equiv \beta(u,t)$. If the set $\{\bar{\beta}(u)\}$ is a set of classical paths $\{q_c(u)\}$ labeled by a parameter u (say a constant of integration), then $S'[\bar{\beta}(u)] = 0$ by definition of $\bar{\beta}(u)$. If we differentiate with respect to u, we get:

$$S''[\bar{\beta}(u)]\,\frac{\partial\bar{\beta}(u)}{\partial u} = 0\,. \qquad (A1)$$

This is the small-disturbance equation with its explicit solution in terms of the classical path: $S''[\bar{\beta}(u)]$ (second functional derivative of the action evaluated at the classical path) yields the small-disturbance operator; $\partial\bar{\beta}(u)/\partial u$ is its explicit solution, called a Jacobi field along the classical path $\bar{\beta}(u)$. Thus, the derivative of a classical solution with respect to a constant of integration is a solution of the small-disturbance equation. Note that if S is derived from a Lagrangian which does not contain the time explicitly, and we take the time derivative of the differential equations resulting from $S'[\bar{\beta}(u)] = 0$, we find that the classical velocity $\partial\bar{\beta}(u)(t)/\partial t$ is also a solution of (A1).

This method of "variation through geodesics" was studied extensively by J. Milnor[17]. The approach was generalized by C. DeWitt-Morette[6c] for arbitrary action functionals, and independently by the author[1] for Lagrangian actions. This method of generating solutions of (A1) was known to Jacobi[12].

Lagrangian Action

Let us consider the Lagrangian action in n dimensions as a specific example:

$$S[q] \equiv \int_{t_a}^{t_b} L\left(q(t), \dot{q}(t), t\right) dt .$$

One can show by straightforward differentiation with respect to u that the linear mapping $S'[\bar{\beta}(u)]$ maps x into

$$S'[\bar{\beta}(u)]x = \int_{t_a}^{t_b} \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right]_{q=\bar{\beta}(u)} (t)\; x^i(t)\, dt \qquad (A2)$$

if $x(t_b) = x(t_a) = 0$ and there are no discontinuities in the momentum $\partial L/\partial \dot{q}^i$. Differentiating once more with respect to u yields:

$$S''[\bar{\beta}(u)] \frac{\partial \bar{\beta}(u)}{\partial u} x$$

$$= \int_T \left(\left\{A_{ij}(t) + B_{ij}(t)\frac{\partial}{\partial t} + C_{ij}(t)\frac{\partial^2}{\partial t^2}\right\} \frac{\partial \beta^j(u,t)}{\partial u}\right) x^i(t)\, dt, \qquad (A3)$$

where

$$\text{(a)}\quad A_{ij}(t) \equiv \frac{\partial^2 L}{\partial q^i \partial q^j} - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\right)$$

$$\text{(b)}\quad B_{ij}(t) \equiv \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)$$

$$\text{(c)}\quad C_{ij}(t) \equiv - \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} . \qquad (A4)$$

Note that the above matrices satisfy the relations:

$$\tilde{C} = C \qquad (B - \dot{C})^{\sim} = -(B - \dot{C})$$

$$B + \tilde{B} = 2\dot{C} \qquad A - \tilde{A} = \dot{B} - \ddot{C} = \tfrac{1}{2}(\dot{B} - \dot{\tilde{B}}).$$

We assume that $C(t)$, the Jacobian of the transformation from the $\dot{q}$'s to the p's, never vanishes, so that a canonical formalism exists.

If $\bar{\beta}(u)$ is a family of classical paths $q_c$, then both sides of (A2) and (A3) are zero for all $x(t)$:

$$\left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right]_{q=q_c} = 0 \qquad (A5)$$

$$\left[A_{ij}(t) + B_{ij}(t)\frac{d}{dt} + C_{ij}(t)\frac{d^2}{dt^2}\right]_{q=q_c} f^j(t) = 0. \qquad (A6)$$

The first equation is the familiar Euler-Lagrange equation, yielding the classical solutions $q_c(t,u)$ where u is any of the 2n constants of integration, or any other parameter (e.g. $t_a$ or $t_b$).

The second equation is the small-disturbance equation, and the bracketed second-order linear differential operator is the (Hermitian) small-disturbance operator. It is solved by $\partial q_c(t,u)/\partial u$.

Attempts at solving (A6) by "frontal assault" are sometimes found in the literature (see, e.g., Reference 8), and usually yield only approximate solutions, if any at all.

A convenient set of solutions is obtained by using endpoint boundary conditions:

$$q_c^i(t_a) = q_a^i \qquad q_c^i(t_b) = q_b^i.$$

Thus, for any fixed $\{i,i'\} = \{1,2,\ldots,n\}$, the two sets:

$$f^j_{(i)}(t) \equiv \frac{\partial q_c^{\,j}(t)}{\partial q_a^{\,i}} \qquad g^j_{(i')}(t) \equiv \frac{\partial q_c^{\,j}(t)}{\partial q_b^{\,i'}}$$

are sets of solutions of (A6) satisfying the obvious boundary conditions:

$$f^j_{(i)}(t_a) = \delta^j_i \qquad g^j_{(i)}(t_a) = 0$$

$$f^j_{(i)}(t_b) = 0 \qquad g^j_{(i)}(t_b) = \delta^j_i$$

We can use these solutions as building blocks for other solutions, which can usually be written as linear combinations of them.

Two other sets of solutions can be obtained by differentiating $q_c(t)$ with respect to $t_a$ or $t_b$:

$$h^j(t) \equiv \frac{\partial q_c^{\,j}(t)}{\partial t_a} \qquad k^j(t) \equiv \frac{\partial q_c^{\,j}(t)}{\partial t_b}$$

They satisfy the boundary conditions:

$$h^j(t_a) = -\dot{q}_c^{\,j}(t_a) \qquad k^j(t_a) = 0$$

$$h^j(t_b) = 0 \qquad k^j(t_b) = -\dot{q}_c^{\,j}(t_b)$$

Proof. The second and third are obvious since the operations, say, "$\partial / \partial t_a$" and "evaluate at $t_b$" commute. The first and fourth are more subtle. The first is derived as follows:

$$h^j(t_a) = -\left[h^j(t_b) - h^j(t_a)\right] = -\int_t^{t_b} \frac{\partial \dot{q}_c^{\,j}(t)}{\partial t_a}\,dt$$

$$= -\frac{\partial}{\partial t_a}\int_{t_a}^{t_b} \dot{q}_c^{\,j}(t)\,dt - \dot{q}_c^{\,j}(t_a)$$

$$= -\frac{\partial}{\partial t_a}\left(q_b^j - q_a^j\right) - \dot{q}_c^{\,j}(t_a) = -\dot{q}_c^{\,j}(t_a) \blacksquare$$

The fourth relation is derived in a similar manner.

Theorem. Let x(t) and y(t) be two solutions of the small disturbance equation (A6) in one dimension. Their Wronskian depends on t only through C(t):

$$W(t) \equiv \dot{x}(t)y(t) - x(t)\dot{y}(t) = \alpha\, C(t_a)\, C^{-1}(t), \qquad \text{(A7)}$$

where $\alpha$ is a constant, and it is assumed that C(t) never vanishes. If $\alpha \neq 0$, x and y are linearly independent.

Proof.

$$\dot{W} = \ddot{x}y - \ddot{y}x = -C^{-1}(Ax + B\dot{x})y + C^{-1}(Ay + B\dot{y})x$$

$$= -BC^{-1}(\dot{x}y - \dot{y}x) = -BC^{-1}W$$

$$\Longrightarrow W(t) = \alpha \exp\left[-\int_{t_a}^{t} B(s)C^{-1}(s)\,ds\right].$$

However, we can see from (A4) that $B = \dot{C}$ in one dimension, and the result follows.

Green Functions

We now study the Green functions $G^{jk}(t,t')$ of the small-disturbance operator, which satisfy:

$$\left[ A_{ij}(t) + B_{ij}(t)\frac{\partial}{\partial t} + C_{ij}(t)\frac{\partial^2}{\partial t^2} \right] G^{jk}(t,t') = \delta_i^{\,k}\,\delta(t-t'), \tag{A8}$$

where A, B, and C are given by (A4) for $q = q_c$. We restrict ourselves to one dimension.

Theorem

The advanced and retarded Green functions are unique and are given by

$$G^-(t,t') = G^+(t',t) = J(t,t')\,Y(t-t') \tag{A9}$$

where $J(t,t')$ is the Jacobi commutator:

$$J(t,t') = \frac{\dfrac{\partial q_c(t)}{\partial \alpha_1}\dfrac{\partial q_c(t')}{\partial \alpha_2} - \dfrac{\partial q_c(t')}{\partial \alpha_1}\dfrac{\partial q_c(t)}{\partial \alpha_2}}{\dfrac{\partial q_c(t_b)}{\partial \alpha_1}\dfrac{\partial p_c(t_b)}{\partial \alpha_2} - \dfrac{\partial p_c(t_b)}{\partial \alpha_1}\dfrac{\partial q_c(t_b)}{\partial \alpha_2}}, \tag{A10}$$

$\alpha_1$ and $\alpha_2$ being any two constants of integration. $t_b$ in the denominator can be replaced by $t_a$.

Proof. We look for the most general Green function of the form $G^-(t,t') = f(t,t')Y(t-t')$. Upon differentiation, and use of the fact that $x\,\delta'(x) = -\delta(x)$, we have:

$$\frac{\partial G^-}{\partial t} = f_{,1}(t,t')Y(t-t') + f(t,t')\,\delta(t-t')$$

$$\frac{\partial^2 G^-}{\partial t^2} = f_{,11}(t,t')Y(t-t') + 2f_{,1}(t,t')\,\delta(t-t') - f(t,t')\frac{\delta(t-t')}{t-t'}$$

$$D_t G^- \equiv \left[A(t) + B(t)\frac{\partial}{\partial t} + C(t)\frac{\partial^2}{\partial t^2}\right] G^-$$

$$= Y(t-t')D_t f(t,t') + \delta(t-t')\left[B(t) + 2C(t)\frac{\partial}{\partial t} - \frac{C(t)}{t-t'}\right] f(t,t'),$$

where $f_{,1}(t,t')$ denotes the derivative with respect to the first argument, evaluated at $(t,t')$.

Thus, $G^-$ is a Green function if $D_t f(t,t') = 0$, i.e. if $f(t,t')$ is a homogeneous solution in t, and if the coefficient of the delta function at $t = t'$ is 1. If we expand about $t = t'$:

$$f(t,t') = f(t,t) + (t-t')\,f_{,1}(t,t) + \frac{1}{2}(t-t')^2 f_{,11}(t,t) + \dots,$$

the second condition gives the boundary conditions on f:

$$\begin{cases} f(t,t) = 0, \text{ since } C(t) \neq 0 \\ f_{,1}(t,t) = C^{-1}(t). \end{cases}$$

If $f(t,t')$ is a solution in t, then we can write

$$f(t,t') = \beta(t')\,x(t) + \gamma(t')\,y(t),$$

where x and y are two linearly independent solutions. If we insert the boundary conditions, and remember (A7), which indicates that $\dot{x}y - x\dot{y} = \alpha C(t_a)C^{-1}(t)$, we have:

$$f(t,t') = \frac{x(t)y(t') - x(t')y(t)}{C(t_a)\left[y(t_a)\dot{x}(t_a) - \dot{y}(t_a)x(t_a)\right]} . \qquad (A11)$$

Let us choose $x(t) = \partial q_c(t)/\partial\alpha_1$ and $y(t) = \partial q_c(t)/\partial\alpha_2$.
Bt definition of C(t), $C(t_a) \equiv -\partial^2 L/\partial\dot{q}_c^2(t_a) = -\partial p_c(t_a)/\partial\dot{q}_c(t_a)$,
so $-C(t_a)\,\partial\dot{q}_c(t_a)/\partial\alpha_1 = \partial p_c(t_a)/\partial\alpha_1$ . Inserting this in (A11) we see that f(t,t') is given by J(t,t') in (A10). Note that our J(t,t') is what Bryce DeWitt[18] calls $\tilde{G}$ (he defines $\tilde{G}$ by $G^+ - G^-$, but since his Green functions are the negative of ours, $\tilde{G} = J$).

The greatest simplification in J(t,t') as expressed in (A10) occurs when the constants of integration are initial (or final) position and momentum, for example $\alpha_1 = q_c(t_a) = q_a$ , $\alpha_2 = p_c(t_a) = p_a$ .
The denominator is then equal to 1, and

$$J(t,t') = \frac{\partial q_c(t)}{\partial q_a}\frac{\partial q_c(t')}{\partial p_a} - \frac{\partial q_c(t')}{\partial q_a}\frac{\partial q_c(t)}{\partial p_a} \equiv \left\{ q_c(t), q_c(t') \right\}_{(q_a,p_a)} .$$

This Poisson bracket becomes the commutator $\left[\underset{\sim}{Q}(t), \underset{\sim}{Q}(t')\right]/i\hbar$ when the system is quantized, whence the name of the function.

Feynman's Green function, which vanishes at $t_a$ and $t_b$, can be built from $G^-$ and $G^+$ as follows:

$$\begin{aligned} G(t,t') &= G^-(t,t') + J(t,t_a)J(t_b,t')/J(t_a,t_b) \\ &= G^+(t,t') + J(t,t_b)J(t_a,t')/J(t_a,t_b). \qquad (A12) \end{aligned}$$

Indeed, it is readily apparent that the additions to $G^-$ and $G^+$ are homogeneous solutions, and that G vanishes when t or t' is $t_a$ or $t_b$. Another form for G is shown in the main text (Equation 13).

Particle in a Potential

Let us concentrate on the case of a particle in a potential in one dimension, with Lagrangian $L = m\dot{q}^2/2 - V(q)$. The dynamical equation is

$$\ddot{q}_c(t) + m^{-1} V'[q_c(t)] = 0 . \qquad (A13)$$

The small-disturbance equation is:

$$\left[ -\frac{d^2}{dt^2} - m^{-1} V''[q_c(t)] \right] f(t) = 0. \qquad (A14)$$

Consider two linearly independent solutions of (A14), D and $\bar{D}$, satisfying:

$$\begin{array}{ll} D(t_b) = 1 & \bar{D}(t_b) = 0 \\ \dot{D}(t_b) = 0 & \dot{\bar{D}}(t_b) = -1 \end{array} \qquad (A15)$$

Their Wronskian $W = D\dot{\bar{D}} - \dot{D}\bar{D}$ is constant and equal to -1. D and $\bar{D}$ depend on $t_b$, $t_a$, $q_b$ and $q_a$ through $q_c(t)$. The antisymmetric Jacobi commutator along the classical path $q_c(t)$ can be shown to be:

$$J(t, t') = \bar{D}(t) D(t') - D(t) \bar{D}(t').$$

It is obviously a solution of (A14) in both t and t'.

Classical path in terms of Jacobi fields

$$q_c(t) = A \int_{t_a}^{t} D(s)\,ds + B \int_{t_a}^{t} \bar{D}(s)\,ds + q_a$$

where

$$A \equiv \frac{q_b - q_a - V'(q_b) \int_T \bar{D}(u)\,du}{\int_T D(s)\,ds} \qquad B \equiv V'(q_b). \qquad (A16)$$

Proof. $\dot{q}_c(t)$, being a derivative of the classical path, is a solution of the small-disturbance equation, and hence a linear combination of D and $\bar{D}$: $\dot{q}_c(t) = AD(t) + B\bar{D}(t)$. Integrating from $t_a$ to t yields:

$$q_c(t) = A\int_{t_a}^{t} D(s)ds + B\int_{t_a}^{t} \bar{D}(s)ds + q_c(t_a).$$

However, $q_c(t)$ is now the solution of a third-order differential equation. Therefore, we need a third boundary condition, other than $q_c(t_a) = q_a$ and $q_c(t_b) = q_b$. It is provided by the dynamical equation (A13) evaluated, say, at $t_b$. This gives A and B. Note that:

$$\dot{q}_c(t_a) = A\,D(t_a) + B\,\bar{D}(t_a) \qquad (A17)$$

$$\dot{q}_c(t_b) = A\,. \qquad (A18)$$

Criterion for non-existence of a classical path

What must the relatioship between $t_a, t_b, q_a, q_b$ be in order for a classical path $q_c(t)$ such that $q_c(t_a) = q_a$, $q_c(t_b) = q_b$ not to exist? The answer is given in terms of Jacobi fields. $q_c$ will not exist if:

$$\left.\begin{array}{ll} & a)\ \int_T D(s)ds = 0,\ \underline{\text{or}}\ \bar{D}(t_a) = M^{-1} = 0 \\ \underline{\text{AND}} & b)\ q_b - q_a - V'(q_b)\int_T \bar{D}(s)ds \neq 0 \end{array}\right\} \qquad (A19)$$

This is easily proved by looking at (A16) which gives $q_c(t)$ in terms of the Jacobi fields. $q_c(t)$ is infinite if the denominator of A is zero (first condition) and the numerator of A is nonzero (second condition). That the two forms of the first condition are equivalent can be seen by differentiating (A17) with respect to $q_b$. On the right hand side, we get $\partial\dot{q}_c(t_a)/\partial q_b = M = 1/\bar{D}(t_a)$, and on the left hand side we get a fraction with denominator $\left(\int_T D(s)ds\right)^2$. Thus, whenever $\bar{D}(t_a)$ vanishes, $\int_T D(s)ds$ must also vanish.

For a general discussion of these conditions in the context of caustics and catastrophe theory, see Ref. 19.

Zero Jacobi field. The only Jacobi field vanishing at both $t_a$ and $t_b$ is $f(t) = 0$, unless

$$\left.\begin{aligned} &\text{a) } \bar{D}(t_a) = M^{-1} = 0 \\ \underline{\text{AND}} \quad &\text{b) } q_b - q_a - V'(q_b)\int_T \bar{D}(s)\,ds = 0 \end{aligned}\right\}, \quad (A20)$$

in which case $f(t) = a\bar{D}(t)$, where a is an arbitrary constant.

Proof. It is obtained by writing $f(t) = a\bar{D}(t)+bD(t)$ and putting in the boundary conditions. However, if $\bar{D}(t_a) = 0$, we may not have a classical path, in which case a Jacobi field is meaningless. Therefore the second condition is necessary to insure that one or more classical paths might exist.

Example: The harmonic oscillator

We illustrate this with the harmonic oscillator $(V(q) = \frac{1}{2}\omega^2 q^2)$. The classical path, for arbitrary endpoints, is given by:

$$q_c(t) = \frac{q_a \sin\omega(t_b - t) + q_b \sin\omega(t - t_a)}{\sin\omega T} = A\cos(\omega t + \varphi)$$

where $T = t_b - t_a$ and

$$\left\{\begin{aligned} A &= (\sin\omega T)^{-1}\left[q_a^2 + q_b^2 - 2q_a q_b \cos\omega T\right]^{1/2} \\ \varphi &= \operatorname{Arc}\cos\left(\frac{q_a \sin\omega t_b - q_b \sin\omega t_a}{\left[q_a^2 + q_b^2 - 2q_a q_b \cos\omega T\right]^{1/2}}\right) \end{aligned}\right.$$

It may fail to exist when $\sin(\omega T) = 0$ (the amplitude becomes infinite), except when $q_a = q_b = 0$, in which case there is an infinite number of $q_c$'s.

The various cases are summarized in Table A1.

The Jacobi fields in this case are:

$$D(t) = \cos\omega(t_b - t) \qquad \bar{D}(t) = \frac{1}{\omega}\sin\omega(t_b - t)$$

We can quickly verify all our criteria. We have:

$$(a) \int_T D(s)\,ds = \bar{D}(t_a) = \frac{1}{\omega}\sin\omega T$$

$$(b) \int_T \bar{D}(s)\,ds = \frac{1}{\omega^2}(1 - \cos\omega T)$$

If $\omega T = n\pi$, we have no classical path, unless:

$$q_b - q_a - V'(q_b)\,\omega^{-2}(1 - \cos\omega T) = 0$$

i.e. if $q_a = q_b$ and $\omega T = 2n\pi$ (yielding one path), or if $q_a = q_b = 0$, which implies that $V'(q_b) = 0$ (yielding an infinite number of paths).

| Harmonic Oscillator | $q_a \neq q_b$ | $q_a = q_b = q_0 \neq 0$ | $q_a = q_b = 0$ |
|---|---|---|---|
| $\omega T \neq n\pi$ | Unique $q_c(t)$ exists and is given by (5.89) | $q_c(t) = q_0 \dfrac{\cos\omega\left(t - \frac{t_a+t_b}{2}\right)}{\cos\left(\frac{\omega T}{2}\right)}$ | $q_c(t) = 0$ |
| $\omega T = 2n\pi$ | $q_c(t)$ never exists | $q_c(t) = q_0 \dfrac{\cos\omega\left(t - \frac{t_a+t_b}{2}\right)}{\cos(n\pi)}$ | Noncountably infinite number of classical paths given by: $q_c(t) = A\sin\omega(t_b - t)$ (A arbitrary) |
| $\omega T = (2n+1)\pi$ | $q_c(t)$ never exists | $q_c(t)$ never exists | |

Table A1. Classical Paths for the Harmonic Oscillator $(n = \ldots,-1,0,1,2,\ldots)$

The Commutator Function

The dynamical equation (A18) can be solved by quadratures: If we substitute $\dot{q}_c(t) = u$, we obtain the energy $E \equiv \frac{1}{2} mu^2 + V(q_c) =$ const. as a first integral. A second integration gives:

$$F(t, t_a, q_c, q_a, E) \equiv t - t_a - \sqrt{\frac{m}{2}} \int_{q_a}^{q_c} \frac{dx}{\sqrt{E - V(x)}} = 0 ,$$

which yields $t(q_c)$ rather than $q_c(t)$. In order to differentiate the classical path with respect to the constants of integration (here, the energy E and the initial position $q_a$), we will need the implicit function theorem. The latter states essentially that:

$$F(x_1, \dots, x_n) = 0 \implies \frac{\partial x_i}{\partial x_j} = - \frac{\partial F / \partial x_j}{\partial F / \partial x_i} \quad (i \neq j).$$

This gives

$$(a) \; \frac{\partial q_c}{\partial q_a} = \sqrt{\frac{E - V(q_c)}{E - V(q_a)}}$$

$$(b) \; \frac{\partial q_c}{\partial E} = \frac{1}{2} \sqrt{E - V(q_c)} \int_{q_a}^{q_c} \left[ E - V(x) \right]^{-3/2} dx \qquad \text{(A21)}$$

$$(c) \; \frac{\partial q_c}{\partial t} = \dot{q}_c(t) = \sqrt{\frac{2}{m} \left[ E - V(q_c) \right]}$$

$$(d) \; \frac{\partial q_c}{\partial t_a} = - \sqrt{\frac{2}{m} \left[ E - V(q_c) \right]}$$

$$(e)\ \frac{\partial \dot{q}_c(t_b)}{\partial E} = \left(\frac{\partial}{\partial E}\frac{\partial q_c}{\partial t}\right)_{t=t_b} = \left\{2m\left[E - V(q_c(t_b))\right]\right\}^{-1/2}$$

$$(f)\ \frac{\partial \dot{q}_c(t_b)}{\partial q_a} = \left(\frac{\partial}{\partial q_a}\frac{\partial q_c}{\partial t}\right)_{t=t_b} = 0$$

Substituting these in (A10), we obtain the commutator (here $p_c = m\dot{q}_c$, $\alpha_1 = q_a$, $\alpha_2 = E$):

$$J(t,t') = \sqrt{\frac{[E-V(q_c(t))][E-V(q_c(t'))]}{2m}} \int_{q_c(t)}^{q_c(t')} \frac{dx}{[E-V(x)]^{3/2}} \qquad (A22)$$

If the constants of integration are initial position and momentum $q_a$ and $p_a$, then $J(t,t')$ is still given by (A22) with E replaced by $p_a^2/2m + V(q_a)$. This is not a trivial statement (compare with (A26) ), as we show below.

Proof. In terms of $q_a$ and $p_a$, the solution is

$$F(t,t_a,q_c,q_a,p_a) \equiv t - t_a - \sqrt{\frac{m}{2}} \int_{q_a}^{q_c} \left[\frac{p_a^2}{2m} + V(q_a) - V(x)\right]^{-1/2} dx = 0 \qquad \text{(A23)}$$

Then

$$\frac{\partial q_c}{\partial q_a} = -\frac{\partial F/\partial q_a}{\partial F/\partial q_c}$$

$$= \left[\frac{p_a^2}{2m} + V(q_a) - V(q_c)\right]^{1/2} \left\{ \frac{\sqrt{2m}}{p_a} + \frac{V'(q_a)}{2} \int_{q_a}^{q_c} dx \left[\frac{p_a^2}{2m} + V(q_a) - V(x)\right]^{-3/2} \right\}$$

$$\frac{\partial q_c}{\partial p_a} = -\frac{\partial F/\partial p_a}{\partial F/\partial q_c}$$

$$= \frac{p_a}{2m} \left\{ \frac{p_a^2}{2m} + V(q_a) - V(q_c) \right\}^{1/2} \int_{q_a}^{q_c} \left\{ \frac{p_a^2}{2m} + V(q_a) - V(x) \right\}^{-3/2} dx \qquad \text{(A24)}$$

Substituting the above in expression (15) for J, some terms cancel out and we get the result. The non-triviality of this result is illustrated by the fact that $\partial q_c / \partial q_a$ in (A24) is <u>not</u> obtained from $\partial q_c / \partial q_a$ in (A21a) by simply replacing E by $p_a^2 / 2m + V(q_a)$.

We can give the commutator in terms of the endpoints $q_a$ and $q_b$. For this we have:

$$\begin{cases} F(t,t_a,q_c,q_a,E) \equiv t-t_a-\sqrt{\frac{m}{2}}\int_{q_a}^{q_c}[E-V(x)]^{-1/2}dx \\ G(t_b,t_a,q_b,q_a,E) \equiv t_b-t_a-\sqrt{\frac{m}{2}}\int_{q_a}^{q_b}[E-V(x)]^{-1/2}dx, \end{cases} \quad (A25)$$

that is, E in the first equation is really a function of $q_a$ and $q_b$, given implicitly by the second equation. It is no longer an independent constant of integration, but $q_b$ is. Thus, we have:

$$\frac{\partial q_c}{\partial q_a} = -\frac{\frac{\partial F}{\partial q_a}}{\frac{\partial F}{\partial q_c}} = -\frac{\sqrt{\frac{m}{2}}\{E-V(q_a)\}^{-1/2}+\frac{1}{2}\sqrt{\frac{m}{2}}\frac{\partial E}{\partial q_a}\int_{q_a}^{q_c}[E-V(x)]^{-3/2}dx}{-\sqrt{\frac{m}{2}}[E-V(q_c)]^{-1/2}}$$

where $\partial E/\partial q_a$ is obtained by using the implicit function theorem on G:

$$\frac{\partial E}{\partial q_a} = -\frac{\partial G/\partial q_a}{\partial G/\partial E} = \frac{-2}{\sqrt{E-V(q_a)}}\left\{\int_{q_a}^{q_c}[E-V(x)]^{-3/2}dx\right\}^{-1}$$

Finally,

$$\frac{\partial q_c}{\partial q_a} = \left[\frac{E-V(q_c)}{E-V(q_a)}\right]^{1/2}\cdot\frac{\int_{q_c}^{q_b}[E-V(x)]^{-3/2}dx}{\int_{q_a}^{q_b}[E-V(x)]^{-3/2}dx}$$

Similarly, we find:

$$\frac{\partial q_c}{\partial q_b} = \left[\frac{E-V(q_c)}{E-V(q_b)}\right]^{1/2}\int_{q_a}^{q_c}[E-V(x)]^{-3/2}dx\Big/\int_{q_a}^{q_b}[E-V(x)]^{-3/2}dx$$

As for the Van Vleck - Morette function $M = -m\,\partial\dot{q}_c(t_b)/\partial q_a$, we use:

$$\dot{q}_c(t_b) = \left[\frac{2}{m}\left\{E(q_a,q_b) - V(q_b)\right\}\right]^{1/2}.$$

This gives:

$$M = -m\frac{\partial \dot{q}_c(t_b)}{\partial q_a} = -\frac{m}{2}\sqrt{\frac{2}{m}}\left[E - V(q_b)\right]^{-1/2}\frac{\partial E}{\partial q_a},$$

i.e.

$$M = \sqrt{\frac{m}{2}}\left\{\left[E-V(q_a)\right]\left[E-V(q_b)\right]\right\}^{-1/2} \Big/ \int_{q_a}^{q_b}\left[E-V(x)\right]^{-3/2}dx.$$

Finally, the commutator in terms of the endpoints is given by (A10) with $x_1 = q_a$ and $x_2 = q_b$:

$$\begin{aligned} J(t,t') = {} & \left(\frac{2}{m}\right)^{1/2}\sqrt{E-V(q_c(t))}\sqrt{E-V(q_c(t'))}\left[\int_{q_a}^{q_b}\{E-V(x)\}^{-3/2}dx\right]^{-1} \\ & \times\left\{\int_{q_c(t)}^{q_b}dx\left[E-V(x)\right]^{-3/2}\int_{q_a}^{q_c(t')}dy\left[E-V(y)\right]^{-3/2} - \right. \\ & \left. - \int_{q_c(t')}^{q_b}dx\left[E-V(x)\right]^{-3/2}\int_{q_a}^{q_c(t)}dy\left[E-V(y)\right]^{-3/2}\right\}, \qquad (A26) \end{aligned}$$

where $q_c(t,q_a,q_b)$ and $E(q_a,q_b)$ are given implicitly by (A25).

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*Department of Mechanical Engineering, University of Maryland.

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*Research supported by the National Science Foundation*

PP 142
Lockman, Robert F., Jehn, Christopher, and Shughart, William F. II, "Models for Estimating Premature Losses and Recruiting District Performance," 36 pp., Dec 1975 (Presented at the RAND Conference on Defense Manpower, Feb 1976; to be published in the conference proceedings) AD A 020 443

PP 143
Horowitz, Stanley and Sherman, Allan (LCdr., USN), "Maintenance Personnel Effectiveness in the Navy," 33 pp., Jan 1976 (Presented at the RAND Conference on Defense Manpower, Feb 1976; to be published in the conference proceedings) AD A021 581

PP 144
Durch, William J., "The Navy of the Republic of China – History, Problems, and Prospects," 66 pp., Aug 1976 (To be published in "A Guide to Asiatic Fleets," ed. by Barry M. Blechman; Naval Institute Press) AD A030 460

PP 145
Kelly, Anne M., "Port Visits and the "Internationalist Mission" of the Soviet Navy," 36 pp., Apr 1976 AD A023 436

PP 146
Palmour, Vernon E., "Alternatives for Increasing Access to Scientific Journals," 6 pp., Apr 1975 (Presented at the 1975 IEEE Conference on Scientific Journals, Cherry Hill, N.C., Apr 28-30; published in IEEE Transactions on Professional Communication, Vol. PC-18, No. 3, Sep 1975) AD A021 798

PP 147
Kessler, J. Christian, "Legal Issues in Protecting Offshore Structures," 33 pp., Jun 1976 (Prepared under task order N00014-68-A-0091-0023 for ONR) AD A028 389

PP 148
McConnell, James M., "Military-Political Tasks of the Soviet Navy in War and Peace," 62 pp., Dec 1975 (Published in Soviet Oceans Development Study of Senate Commerce Committee October 1976) AD A022 590

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Squires, Michael L., "Counterforce Effectiveness: A Comparison of the Tsipis "K" Measure and a Computer Simulation," 24 pp., Mar 1976 (Presented at the International Study Association Meetings, 27 Feb 1976) AD A022 591

PP 150
Kelly, Anne M. and Petersen, Charles, "Recent Changes in Soviet Naval Policy: Prospects for Arms Limitations in the Mediterranean and Indian Ocean," 28 pp., Apr 1976, AD A 023 723

PP 151
Horowitz, Stanley A., "The Economic Consequences of Political Philosophy," 8 pp., Apr 1976 (Reprinted from Economic Inquiry, Vol. XIV, No. 1, Mar 1976)

PP 152
Mizrahi, Maurice M., "On Path Integral Solutions of the Schrodinger Equation, Without Limiting Procedure,"* 10 pp., Apr 1976 (Reprinted from Journal of Mathematical Physics, Vol. 17, No. 4 (Apr 1976), 566-575).
*Research supported by the National Science Foundation*

PP 153
Mizrahi, Maurice M., "WKB Expansions by Path Integrals, With Applications to the Anharmonic Oscillator,"* 137 pp., May 1976, AD A025 440
*Research supported by the National Science Foundation*

PP 154
Mizrahi, Maurice M., "On the Semi-Classical Expansion in Quantum Mechanics for Arbitrary Hamiltonians," 19 pp., May 1976 (Published in Journal of Mathematical Physics, Vol. 18, No. 4, p. 786, Apr 1977), AD A025 441

PP 155
Squires, Michael L., "Soviet Foreign Policy and Third World Nations," 26 pp., Jun 1976 (Prepared for presentation at the Midwest Political Science Association meetings, Apr 30, 1976) AD A028 388

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Stallings, William, "Approaches to Chinese Character Recognition," 12 pp., Jun 1976 (Reprinted from Pattern Recognition (Pergamon Press), Vol. 8, pp. 87-98, 1976) AD A028 692

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Morgan, William F., "Unemployment and the Pentagon Budget: Is There Anything in the Empty Pork Barrel?" 20 pp., Aug 1976 AD A030 455

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Haskell, LCdr. Richard D. (USN), "Experimental Validation of Probability Predictions," 25 pp., Aug 1976 (Presented at the Military Operations Research Society Meeting, Fall 1976) AD A030 458

PP 159
McConnell, James M., "The Gorshkov Articles, The New Gorshkov Book and Their Relation to Policy," 93 pp., Jul 1976 (Published in *Soviet Naval Influence: Domestic and Foreign Dimensions*, ed. by M. McCGwire and J. McDonnell; New York; Praeger, 1977) AD A029 227

PP 160
Wilson, Desmond P., Jr., "The U.S. Sixth Fleet and the Conventional Defense of Europe," 50 pp., Sep 1976 (Submitted for publication in Adelphi Papers, I.I.S.S., London) AD A030 457

PP 161
Melich, Michael E. and Peet, Vice Adm. Ray (USN, Retired), "Fleet Commanders: Afloat or Ashore?" 9 pp., Aug 1976 (Reprinted from *U.S. Naval Institute Proceedings*, Jun 1976) AD A030 456

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Friedheim, Robert L., "Parliamentary Diplomacy," 106 pp. Sep 1976 AD A033 306

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Lockman, Robert F., "A Model for Predicting Recruit Losses," 9 pp., Sep 1976 (Presented at the 84th annual convention of the American Psychological Association, Washington, D.C., 4 Sep 1976) AD A030 459

PP 164
Mahoney, Robert B., Jr., "An Assessment of Public and Elite Perceptions in France, The United Kingdom, and the Federal Republic of Germany, 31 pp., Feb 1977 (Presented at Conference " Perception of the U.S. – Soviet Balance and the Political Uses of Military Power" sponsored by Director, Advanced Research Projects Agency, April 1976) AD 036 599

PP 165
Jondrow, James M. "Effects of Trade Restrictions on Imports of Steel," 67 pp., November 1976, (Delivered at ILAB Conference in Dec 1976)

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Feldman, Paul, "Impediments to the Implementation of Desirable Changes in the Regulation of Urban Public Transportation," 12 pp., Oct 1976, AD A033 322

PP 166 – Revised
Feldman, Paul, "Why It's Difficult to Change Regulation," Oct 1976

PP 167
Kleinman, Samuel, "ROTC Service Commitments: a Comment," 4 pp., Nov 1976, (To be published in *Public Choice*, Vol. XXIV, Fall 1976) AD A033 305

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Lockman, Robert F., "Revalidation of CNA Support Personnel Selection Measures," 36 pp., Nov 1976

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Jacobson, Louis S., "Earnings Losses of Workers Displaced from Manufacturing Industries," 30 pp., Nov 1976, (Delivered at ILAB Conference in Dec 1976), AD A039 809

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Brechling, Frank P., "A Time Series Analysis of Labor Turnover," Nov 1976. (Delivered at ILAB Conference in Dec 1976)

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Ralston, James M., "A Diffusion Model for GaP Red LED Degradation," 10 pp., Nov 1976, (Published in Journal of Applied Pysics, Vol. 47, pp. 4518-4527, Oct 1976)

PP 172
Classen, Kathleen P., "Unemployment Insurance and the Length of Unemployment," Dec 1976, (Presented at the University of Rochester Labor Workshop on 16 Nov 1976)

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Kleinman, Samuel D., "A Note on Racial Differences in the Added-Worker/Discouraged-Worker Controversy," 2 pp., Dec 1976, (Published in the American Economist, Vol. XX, No. 1, Spring 1976)

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Mahoney, Robert B., Jr., "A Comparison of the Brookings and International Incidents Projects," 12 pp. Feb 1977 AD 037 206

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Levine, Daniel; Stoloff, Peter and Spruill, Nancy, "Public Drug Treatment and Addict Crime," June 1976, (Published in Journal of Legal Studies, Vol. 5, No. 2)

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Felix, Wendi, "Correlates of Retention and Promotion for USNA Graduates," 38 pp., Mar 1977, AD A039 040

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Lockman, Robert F. and Warner, John T., "Predicting Attrition: A Test of Alternative Approaches," 33 pp. Mar 1977. (Presented at the OSD/ONR Conference on Enlisted Attrition Xerox International Training Center, Leesburg, Virginia, 4-7 April 1977), AD A039 047

PP 178
Kleinman, Samuel D., "An Evaluation of Navy Unrestricted Line Officer Accession Programs," 23 pp. April 1977, (To be presented at the NATO Conference on Manpower Planning and Organization Design, Stresa, Italy, 20 June 1977), AD A039 048

PP 179
Stoloff, Peter H. and Balut, Stephen J., "Vacate: A Model for Personnel Inventory Planning Under Changing Management Policy," 14 pp. April 1977, (Presented at the NATO Conference on Manpower Planning and Organization Design, Stresa, Italy, 20 June 1977), AD A039 049

PP 180
Horowitz, Stanley A. and Sherman, Allan, "The Characteristics of Naval Personnel and Personnel Performance," 16 pp. April 1977, (Presented at the NATO Conference on Manpower Planning and Organization Design, Stresa, Italy, 20 June 1977), AD A039 050

PP 181
Balut, Stephen J. and Stoloff, Peter, "An Inventory Planning Model for Navy Enlisted Personnel," 35 pp., May 1977, (Prepared for presentation at the Joint National Meeting of the Operations Research Society of America and The Institute for Management Science. 9 May 1977, San Francisco, California), AD A042 221

PP 182
Murray, Russell, 2nd, "The Quest for the Perfect Study or My First 1138 Days at CNA," 57 pp., April 1977

PP 183
Kassing, David, "Changes in Soviet Naval Forces," 33 pp., November, 1976, (Published as part of Chapter 3, "General Purpose Forces: Navy and Marine Corps," in Arms, Men, and Military Budgets, Francis P. Hoeber and William Schneider, Jr. (eds.), (Crane, Russak & Company, Inc.: New York), 1977), AD A040 106

PP 184
Lockman, Robert F., "An Overview of the OSD/ONR Conference on First Term Enlisted Attrition," 22 pp., June 1977, (Presented to the 39th MORS Working Group on Manpower and Personnel Planning, Annapolis, Md., 28-30 June 1977), AD A043 618

PP 185
Kassing, David, "New Technology and Naval Forces in the South Atlantic," 22 pp. (This paper was the basis for a presentation made at the Institute for Foreign Policy Analyses, Cambridge, Mass., 28 April 1977), AD A043 619

PP 186
Mizrahi, Maurice M., "Phase Space Integrals, Without Limiting Procedure," 31 pp., May 1977, (Invited paper presented at the 1977 NATO Institute on Path Integrals and Their Application in Quantum Statistical, and Solid State Physics, Antwerp, Belgium, July 17-30, 1977) (Published in Journal of Mathematical Physics 19(1), p. 298, Jan 1978), AD A040 107

PP 187
Coile, Russell C., "Nomography for Operations Research," 35 pp., April 1977, (Presented at the Joint National Meeting of the Operations Research Society of America and The Institute for Management Services, San Francisco, California, 9 May 1977), AD A043 620

PP 188
Durch, William J., "Information Processing and Outcome Forecasting for Multilateral Negotiations: Testing One Approach," 53 pp., May 1977 (Prepared for presentation to the 18th Annual Convention of the International Studies Association, Chase-Park Plaza Hotel, St. Louis, Missouri, March 16-20, 1977), AD A042 222

PP 189
Coile, Russell C., "Error Detection in Computerized Information Retrieval Data Bases," July, 1977, 13 pp. Presented at the Sixth Cranfield International Conference on Mechanized Information Storage and Retrieval Systems, Cranfield Institute of Technology, Cranfield, Bedford, England, 26-29 July 1977, AD A043 580

PP 190
Mahoney, Robert B., Jr., "European Perceptions and East-West Competition," 96 pp., July 1977 (Prepared for presentation at the annual meeting of the International Studies Association, St. Louis, Mo., March, 1977), AD A043 661

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Sawyer, Ronald, "The Independent Field Assignment: One Man's View," August 1977, 25 pp.

PP 192
Holen, Arlene, "Effects of Unemployment Insurance Entitlement on Duration and Job Search Outcome," August 1977, 6 pp., (Reprinted from Industrial and Labor Relations Review, Vol., 30, No. 4, Jul 1977)

PP 193
Horowitz, Stanley A., "A Model of Unemployment Insurance and the Work Test," August 1977, 7 pp. (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 40, Jul 1977)

PP 194
Classen, Kathleen P., "The Effects of Unemployment Insurance on the Duration of Unemployment and Subsequent Earnings," August 1977, 7 pp. (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 40, Jul 1977)

PP 195
Brechling, Frank, "Unemployment Insurance Taxes and Labor Turnover: Summary of Theoretical Findings," 12 pp. (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 40, Jul 1977)

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Ralston, J. M. and Lorimor, O. G., "Degradation of Bulk Electroluminescent Efficiency in Zn, O-Doped GaP LED's," July 1977, 3 pp. (Reprinted from IEEE Transactions on Electron Devices, Vol. ED-24, No. 7, July 1977)

PP 197
Wells, Anthony R., "The Centre for Naval Analyses," 14 pp., Dec 1977, AD A049 107

PP 198
Classen, Kathleen P., "The Distributional Effects of Unemployment Insurance," 25 pp., Sept. 1977 (Presented at a Hoover Institution Conference on Income Distribution, Oct 7-8, 1977)

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Durch, William J., "Revolution From A F.A.R. – The Cuban Armed Forces in Africa and the Middle East," Sep 1977, 16 pp., AD A046 268

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Powers, Bruce F., "The United States Navy," 40 pp. Dec 1977. (To be published as a chapter in The U.S. War Machine by Salamander Books in England during 1978), AD A049 108

PP 201
Durch, William J., "The Cuban Military in Africa and The Middle East: From Algeria to Angola," Sep 1977, 67 pp., AD A045 675

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Feldman, Paul, "Why Regulation Doesn't Work," (Reprinted from *Technological Change and Welfare in the Regulated Industries and Review of Social Economy*, Vol. XXIX, March, 1971, No. 1.) Sep 1977, 8 pp.

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Feldman, Paul, "Efficiency, Distribution, and the Role of Government in a Market Economy," (Reprinted from *The Journal of Political Economy*, Vol. 79, No. 3, May/June 1971.) Sep 1977, 19 pp., AD A045 675

PP 204
Wells, Anthony R., "The 1967 June War: Soviet Naval Diplomacy and The Sixth Fleet – A Reappraisal," Oct 1977, 36 pp., AD A047 236

PP 205
Coile, Russell C., "A Bibliometric Examination of the Square Root Theory of Scientific Publication Productivity," (Presented at the annual meeting of the American Society for Information Science, Chicago, Illinios, 29 September 1977.) Oct 1977, 6 pp., AD A047 237

PP 206
McConnell, James M., "Strategy and Missions of the Soviet Navy in the Year 2000," 48 pp., Nov 1977, (Presented at a Conference on Problems of Sea Power as we Approach the 21st Century, sponsored by the American Enterprise Institute for Public Policy Research, 6 October 1977, and subsequently published in a collection of papers by the Institute), AD A047 244

PP 207
Goldberg, Lawrence, "Cost-Effectiveness of Potential Federal Policies Affecting Research & Development Expenditures in the Auto, Steel and Food Industries," 36 pp., Oct 1977, (Presented at Southern Economic Association Meetings beginning 2 November 1977)

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Roberts, Stephen S., "The Decline of the Overseas Station Fleets: The United States Asiatic Fleet and the Shanghai Crisis, 1932," 18 pp., Nov 1977, (Reprinted from The American Neptune, Vol. XXXVII., No. 3, July 1977), AD A047 245

PP 209 – Classified.

PP 210
Kassing, David, "Protecting The Fleet," 40 pp., Dec 1977 (Prepared for the American Enterprise Institute Conference on Problems of Sea Power as We Approach the 21st Century, October 6-7, 1977), AD A049 109

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Mizrahi, Maurice M., "On Approximating the Circular Coverage Function," 14 pp., Feb 1978

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Mangel, Marc, "On Singular Characteristic Initial Value Problems with Unique Solutions," 20 pp., Jun 1978 (To be submitted for publication in Journal of Mathematical Analysis and Its Applications)

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Mangel, Marc, "Fluctuations in Systems with Multiple Steady States. Application to Lanchester Equations," 12 pp., Feb 78, (Presented at the First Annual Workshop on the Information Linkage Between Applied Mathematics and Industry, Naval PG School, Feb 23-25, 1978)

PP 214
Weinland, Robert G., "A Somewhat Different View of The Optimal Naval Posture,"37 pp., Jun 1978 (Presented at the 1976 Convention of the American Political Science Association (APSA/IUS Panel on "Changing Strategic Requirements and Military Posture"), Chicago, Ill., September 2, 1976)

PP 215
Coile, Russell C., "Comments on: *Principles of Information Retrieval* by Manfred Kochen, 10 pp., Mar 78, (Published as a Letter to the Editor, Journal of Documentation, Vol. 31, No. 4, pages 298-301, December 1975)

PP 216
Coile, Russell C., "Lotka's Frequency Distribution of Scientific Productivity," 18 pp., Feb 1978, (Published in the Journal of the American Society for Information Science, Vol. 28, No. 6, pp. 366-370, November 1977)

PP 217
Coile, Russell C., "Bibliometric Studies of Scientific Productivity," 17 pp., Mar 78, (Presented at the Annual meeting of the American Society for Information Science held in San Francisco, California, October 1976.)

PP 218 – Classified.

PP 219
Huntzinger, R. LaVar, "Market Analysis with Rational Expectations: Theory and Estimation," 60 pp., Apr 78 (To be submitted for publication in Journal of Econometrics)

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Maurer, Donald E., "Diagonalization by Group Matrices," 26 pp., Apr 78

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Weinland, Robert G., "Superpower Naval Diplomacy in the October 1973 Arab-Israeli War," 76 pp., Jun 1978

PP 222
Mizrahi, Maurice M., "Correspondence Rules and Path Integrals," 30 pp., Jun 1978 (Invited paper presented at the CNRS meeting on "Mathematical Problems in Feynman's Path Integrals," Marseille, France, May 22-26, 1978)

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Mangel, Marc, "Stochastic Mechanics of Molecule-Ion Molecule Reactions," 21 pp., Jun 1978 (To be submitted for publication in Journal of Mathematical Physics)

PP 224
Mangel, Marc, "Aggregation, Bifurcation, and Extinction In Exploited Animal Populations*," 48 pp., Mar 1978 (To be submitted for publication in American Naturalist)
**Portions of this work were started at the Institute of Applied Mathematics and Statistics, University of British Columbia, Vancouver, B.C., Canada*

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Mangel, Marc, "Oscillations, Fluctuations, and the Hopf Bifurcation*," 43 pp., Jun 1978
**Portions of this work were completed at the Institute of Applied Mathematics and Statistics, University of British Columbia, Vancouver, Canada.*

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Ralston, J. M. and J. W. Mann*, "Temperature and Current Dependence of Degradation in Red-Emitting GaP LEDs," 34 pp., Jun 1978

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Mangel, Marc, "Uniform Treatment of Fluctuations at Critical Points," 50 pp., May 1978 (To be submitted for publication in Journal of Statistical Physics)

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Mangel, Marc, "Relaxation at Critical Points: Deterministic and Stochastic Theory," 54 pp., Jun 1978 (To be submitted for publication in Journal of Mathematical Physics)

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Mangel, Marc, "Diffusion Theory of Reaction Rates, I: Formulation and Einstein-Smoluchowski Approximation," 50 pp., Jan 1978

PP 230
Mangel, Marc, "Diffusion Theory of Reaction Rates, II Ornstein-Uhlenbeck Approximation, 34 pp., Feb 1978

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Wilson, Desmond P., Jr., "Naval Projection Forces: The Case for a Responsive MAF," Aug 1978

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Jacobson, Louis, "Can Policy Changes be Made Acceptable to Labor?" Aug 1978 (To be submitted for publication in Industrial and Labor Relations Review)

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Jondrow, James and Levy, Robert A., "Does Federal Expenditure Displace State and Local Expenditure: The Case of Construction Grants," 18 pp., Oct 1978 (To be submitted for publication in Journal of Public Economics)

PP 235
Mizrahi, Maurice M., "The Semiclassical Expansion of the Anharmonic-Oscillator Propagator," 41 pp., Oct 1978 (To be published in the Journal of Mathematical Physics)